# Arkhipov - Bezrukavnikov (continued)

## Last time

already construct a functor $\widetilde{F}: Coh^{G^\vee \times T^\vee}_{free}(\widehat{\widetilde{N}}_{aff}) \to \mathcal{A}$.

where $\widetilde{N} \subset G^\vee/B^\vee \times \mathfrak{g}^\vee$, $\widehat{\widetilde{N}} \subset G^\vee/U^\vee \times \mathfrak{g}^\vee$, $\widehat{\widetilde{N}}_{aff} \subset \overline{G^\vee/U^\vee} \times \mathfrak{g}^\vee$.

$\partial \widehat{\widetilde{N}} = \widehat{\widetilde{N}}_{aff} \setminus \widehat{\widetilde{N}} = \widehat{\widetilde{N}}_{aff} \cap (\overline{G^\vee/U^\vee} \setminus (G^\vee/U^\vee) \times \mathfrak{g}^\vee)$.

$\widetilde{F}(V \otimes \mathcal{O}) = Z(V)$, $\widetilde{F}(\mathcal{O}(\lambda)) = J_\lambda$ for $\lambda \in X^\vee$.

$G^\vee/B^\vee$ is the multi-Proj scheme of $X^\vee$-graded algebra $\mathcal{O}(G^\vee/U^\vee)$

Hence $\partial \widehat{\widetilde{N}}$ contains the support of coker $B_\lambda$ and equals to it if $\lambda$ is regular.

Furthermore, any sheaf set theoretically supported on $\overline{G^\vee/U^\vee} \setminus (G^\vee/U^\vee)$ is scheme-theoretically supported in cok $B_\lambda$ for $\lambda \gg 0$.

## Factor to F

$\widetilde{F}$ extends to $K^b(Coh^{G^\vee \times T^\vee}_{free}(\widehat{\widetilde{N}}_{aff})) \to D^b(\mathcal{A})$.

Let $K_\lambda$ be the Koszul complex associated to $B_\lambda$, i.e., the free resolution of cok $B_\lambda$:

$$0 \to \Lambda^d(V_\lambda) \otimes \mathcal{O} \to \Lambda^{d-1}(V_\lambda) \otimes \mathcal{O}(\lambda) \to \cdots \to V_\lambda \otimes \mathcal{O}((d-1)\lambda) \to \mathcal{O}(d\lambda) \to 0$$

Then $\widetilde{F}(K_\lambda) = 0$, since the image unger gr being the Koszul complex associated to $V_\lambda \twoheadrightarrow \lambda$ is zero.

For $\mathcal{F} \in K^b(Coh^{G^\vee \times T^\vee}_{free}(\widehat{\widetilde{N}}_{aff}))$, whose cohomologies are set-support in $\partial \widehat{\widetilde{N}}$, then it can be represented by a complex $C$, whose components are in $Coh_{\partial \widehat{\widetilde{N}}}(\widehat{\widetilde{N}}_{aff})$. Pick $\lambda$ such that the supports of $C$ are in cok $B_\lambda$, thus $B_\lambda \otimes id_C = 0$, showing $\mathcal{F}$ is a direct summand of $K_\lambda \otimes \mathcal{F}$.

**Lemma** Let $A$ be finitely generated $\mathbb{Z}^N_{\geq 0}$-graded algebra, $X$ be the corresponding multi-Proj scheme.

Let $D^{fr}_A$ denote $K^b(A\text{-mod}_{free})$, $D^{fr,0}_A$ denote the full subcategory whose localization to $D^b(Coh(X))$ is zero.

Then $D^{fr}_A / D^{fr,0}_A$ is identified with a full subcat of $D^b(Coh(X))$.

Proof. For any $C \in D_A^{fr}$, $\lambda_0 \in \mathbb{Z}^N$, there exists $C' \in D_A^{fr}$ and $f: C \to C'$ such that $\mathrm{cone}(f) \in D_A^{fr,0}$, and $(C')^i$ is a sum of modules of the form $A \otimes V(\lambda)$. $\lambda_0 - \lambda \in \mathbb{Z}_+^N$:

Pick $V \subseteq A_\mu$, $\mu \gg 0$ such that $A/V\cdot A$ is supported on $\partial$.

Consider $K = (0 \to \Lambda^d(V) \otimes A(-d\mu) \to \cdots \to V \otimes A(-\mu) \to A \to 0)$

we have $K \otimes C \in D_A^{fr,0}$ and $\mathrm{cone}(K \otimes C \to C)$ satisfy requirement.

Now, for $B \in D_A^{fr}$, pick $B(-\lambda_0)$ dominant, hence

$\mathrm{Hom}_{D_A}(A(\lambda), B) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Coh(X))}(\mathcal{O}(\lambda), \tilde{B})$ for $\lambda_0 - \lambda \in \mathbb{Z}_+^N$.

Hence $\mathrm{Hom}_{D_A^{fr}}(C', B) \xrightarrow{\sim} \mathrm{Hom}_{D^b(Coh(X))}(\tilde{C'}, \tilde{B})$. $\square$

Lemma $\mathcal{O}(\lambda)$ for $\lambda \in X^\vee$ generates $D^{G^\vee}(\tilde{N})$.

$\tilde{N}$ smooth, hence $D^{G^\vee}(\tilde{N})$ is represented by vector bundles.

It suffices to show every $G^\vee$-equivariant bundle $\mathcal{E}$ is filtered by $\mathcal{O}(\lambda)$. Using $Coh^{G^\vee}(\tilde{N}) = Coh^{B^\vee}(n^\vee)$, take the lowest weight of $\Gamma(\mathcal{E}|_{n^\vee})$ to get an injection. $\square$

Conclusion we get $K^b(Coh^{G^\vee \times T^\vee}_{free}(\hat{N}_{aff}))/K^b_0 \to D^{G^\vee \times T^\vee}(\hat{N}) = D^{G^\vee}(\tilde{N})$ is an equivalence of category and obtain $F: D^{G^\vee}(\tilde{N}) \to D(A)$.

The functor $Av_\chi: {}^fP_I \to P_{IW}$

$Av_\chi(L_w) = 0 \Leftrightarrow w \notin {}^fW$

If $w \in {}^fW$, then $\ell(w_0 w) = \ell(w_0) + \ell(w)$, thus $m: Fl \tilde{\times} Fl \to Fl$ restricts to the generic point of $\overline{Fl^\circ \tilde{\times} Fl_w}$ is an isomorphism, hence $\Delta_0 * L_w \neq 0$.

If $w \notin {}^fW$, there exists a root $\alpha$ such that $F \mapsto F * L_w$ factor through $\pi_{\alpha *}$, but $\pi_{\alpha *}(\Delta_0) = 0$.

$Av_\chi(j_{w!}) = \Delta_{w'}$, $Av_\chi(j_{w*}) = \nabla_{w'}$, where $w = w_f w'$, $w_f \in W_f$, $w' \in {}^fW$.

Note $\Delta_0 = \nabla_0$ since $\chi$ is nontrivial on $\overline{Fl^\circ} \setminus Fl^\circ$.

For the first, $w \in {}^fW$, the result is clear since $m$ is isom.

the other follows from $j_{w!} = j_{w_f!} * j_{w'!}$ and $j_{w_f!}/\delta_e$ has the filtration of $L_w$, $w \in W_f \setminus \{e\}$.

For triangulated category $D$ and subset $S$, let $\langle S \rangle$ be the set of objects obtained by extension. Thus

$$Ob(P_1) = \langle j_{w!}[i], i \geq 0 \rangle \cap \langle j_{w*}[i], i \leq 0 \rangle$$

$$Ob(P_{Iw}) = \langle \Delta_w[i], i \geq 0 \rangle \cap \langle \nabla_w[i], i \leq 0 \rangle$$

Hence $Av_\chi(P_1) \subseteq P_{Iw}$, i.e., $Av_\chi$ is exact.

Then get the functor $F_{Iw} : D^{G^\vee}(\tilde{N}) \xrightarrow{F} D(A) \xrightarrow{Av_\chi} D(P_{Iw}) = D_{Iw}$

For $\lambda \in X^\vee$, define $x(\lambda) \in {}^f W$ the corresponding element.

Hence $F_{Iw}(\mathcal{O}(\lambda)) = \Delta_0 * J_\lambda$ is supported in $\overline{Fl^{x(\lambda)}}$.

Furthermore, the restriction to $Fl^{x(\lambda)}$ has rank 1, $\Rightarrow F_{Iw}(\mathcal{O}(\lambda))$ generate $D_{Iw}$

Thus it suffices to show $F_{Iw}$ is fully faithful.

Using the acyclicity of the Koszul complex, one can show $D^{G^\vee}(\tilde{N})$ is also generated by $\mathcal{O}(-\mu) \otimes V_\lambda$, for $\lambda, \mu \in X^\vee_+$.

Using the two generation result, it suffices to show

$$\mathrm{Hom}^\bullet_{D^{G}(\tilde{N})}(V_\lambda \otimes \mathcal{O}, \mathcal{O}(\mu)) = \mathrm{Hom}^\bullet_{D_{Iw}}(\Delta_0 * Z_\lambda, \Delta_0 * J_\mu = \nabla_{x(\mu)}), \lambda, \mu \in X^\vee_+.$$

the former equals to $\mathrm{Hom}_G(V_\lambda, H^\bullet(\tilde{N}, \mathcal{O}(\mu)))$

It is zero if $\bullet > 0$, and the dimension equals to $[\mu : V_\lambda]$.

the latter equals to the stalk at $x(\mu)$ of $\Delta_0 * Z_\lambda$.

Thus one needs to calculate $\dim \mathrm{stalk}_\mu(\Delta_0 * Z_\lambda)$ and show the map for Hom is injective.

Proof of injectivity

Let $D_1^{\neq 0}$ be the thick subcat generated by $L_w$, $\tilde{l}(w) \neq 0$.

$D_1^0 = D_1 / D_1^{\neq 0}$, $P_1^0 = P_1 / P_1 \cap D_1^{\neq 0}$. Since $D_1^{\neq 0} * D_1 \subseteq D_1^{\neq 0}$, $*$ descent to $D_1^0$.

The irreducible objects in $P_1^0$ corresponds to $\pi_1(G)$ and admits a monoidal structure from $D_1^0$. Then $F_0 : Rep(G^\vee) \to P_1 \to P_1^0$ is a monoidal central functor, and $M$ is a tensor endomorphism.

From a general fact, $F_0$ pass through a functor $Rep(G^\vee) \to Rep(H)$, by considering $F_0(\mathcal{O}(G^\vee))$ quoted by its maximal ideal.

$M$ corresponds to a nilpotent element $N_0 \in \mathcal{N}$, then $H \subseteq Z_{G^\vee}(N_0)$.

the largest cell $\Rightarrow N_0 \in \mathcal{N}^0$ the open orbit.

In conclusion we get the functors equal to $F_0$:

$$\mathrm{Coh}^{G^\vee}(\tilde{N}) \xhookrightarrow[\text{restrict to open dense}]{} \mathrm{Coh}^{G^\vee}(N^\circ) = \mathrm{Rep}(Z_{G^\vee}(N^\circ)) \xhookrightarrow[\text{restrict to subgroup}]{} \mathrm{Rep}(H) \hookrightarrow P_1^\circ$$

To show the injectivity of $F_{IW}$, embed $\mathcal{O}(\mu)$ into $V\otimes\mathcal{O}$,

$$\begin{array}{ccc} \mathrm{Hom}(V_\lambda\otimes\mathcal{O}, \mathcal{O}(\mu)) & \longrightarrow & \mathrm{Hom}(F_{IW}(V_\lambda\otimes\mathcal{O}), F_{IW}(\mathcal{O}(\mu))) \\ \downarrow & & \downarrow \\ \mathrm{Hom}(V_\lambda\otimes\mathcal{O}, V\otimes\mathcal{O}) & \longrightarrow & \mathrm{Hom}(F_{IW}(V_\lambda\otimes\mathcal{O}), F_{IW}(V\otimes\mathcal{O})) \end{array}$$

it suffices to show the bottom arrow is injective.

Since $Av_\psi$ does not kill any $L_w$, $w\in {}^fW$, hence $l(w)=0$, any morphism under $Av_\psi$ is non-zero $\Rightarrow$ its image in $P_1^\circ$ is non-zero.

Calculate the dimension of stalk

In the K-group of $D_I$, we know $[j_{s*}]=[j_{s!}]$, $[j_{w_1!} * j_{w_2!}]=[j_{w_1!}]\cdot[j_{w_2!}]$, Hence $[J_\lambda]=[j_{\lambda!}]$. Furthermore $[Z(V)] = \sum_\mu [\mu:V]\cdot[J_\mu]$ shows

$$[F_{IW}(V\otimes\mathcal{O})] = \sum_\mu [\mu:V]\cdot[\Delta_0 * j_{\mu!}] = \sum_\mu [\mu:V]\cdot[\Delta_{x(\mu)}].$$

Thus it suffices to show these stalks concentrate in degree 0, i.e., the objects $F_{IW}(V\otimes\mathcal{O})$ has a filtration with subquotient $\Delta_{x(\mu)}$.

Lemma 1: if it holds for $V_1$, $V_2$, then also for $V_1\otimes V_2$.

Lemma 2: it holds for (quasi-)minisule $V_\lambda$.

TO BE CONTINUED!